

# Sum Formula of Multiple Hurwitz-Zeta Values

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## Abstract

Let  $s_1, \dots, s_d$  be  $d$  positive integers and define the multiple  $t$ -values of depth  $d$  by

$$t(s_1, \dots, s_d) = \sum_{n_1 > \dots > n_d \geq 1} \frac{1}{(2n_1 - 1)^{s_1} \dots (2n_d - 1)^{s_d}},$$

which is equal to the multiple Hurwitz-zeta value  $2^{-w}\zeta(s_1, \dots, s_d; -\frac{1}{2}, \dots, -\frac{1}{2})$  where  $w = s_1 + \dots + s_d$  is called the weight. For  $d \leq n$ , let  $T(2n, d)$  be the sum of all multiple  $t$ -values with even arguments whose weight is  $2n$  and whose depth is  $d$ . Recently Shen and Cai gave formulas for  $T(2n, d)$  for  $d \leq 5$  in terms of  $t(2n)$ ,  $t(2)t(2n-2)$  and  $t(4)t(2n-4)$ . In this short note we generalize Shen-Cai's results to arbitrary depth by using the theory of symmetric functions established by Hoffman.

## 1 Introduction

In recent years multiple zeta functions and many different variations and generalizations have been studied intensively due to their close relations to other objects in a lot of diverse branches of mathematics and physics. In particular, a large number of identities are establishes between their special values. In [4] Shen and Cai found a few very interesting equations which are similar in nature to Euler's identity of double zeta values. They gave formulas of the sum  $E(2n, d)$  of multiple zeta values at even arguments of fixed depth  $d$  and weight  $2n$ , for  $d \leq 4$ . These have been generalized to arbitrary depth by Hoffman [1]. In [3] Shen and Cai turned to the following values

$$t(s_1, \dots, s_d) = \sum_{n_1 > \dots > n_d \geq 1} \frac{1}{(2n_1 - 1)^{s_1} \dots (2n_d - 1)^{s_d}},$$

which we call multiple  $t$ -values of depth  $d$  in this note. It is clear that this is equal to  $2^{-w}\zeta(s_1, \dots, s_d; -\frac{1}{2}, \dots, -\frac{1}{2})$  where  $w = s_1 + \dots + s_d$  is called the weight. Put

$$T(2n, d) = \sum_{\substack{j_1 + \dots + j_d = n \\ j_1, \dots, j_d \geq 1}} t(2j_1, \dots, 2j_d).$$

Using similar but more complicated ideas from [4] Shen and Cai gave a few sum formulas for  $T(2n, d)$  for  $d \leq 5$  in [3]. For example,

$$T(2n, 5) = \frac{7}{128}t(2n) - \frac{3}{64}t(2)t(2n-2) + \frac{1}{320}t(4)t(2n-4). \quad (1)$$

In this note, we shall generalize these to arbitrary depth using ideas from [1] where Hoffman applied the theory of symmetric functions to study the generating function of  $E(2n, d)$ . It turns out that we need both Bernoulli numbers  $B_j$  and Euler numbers  $E_j$  defined by the following generating functions respectively:

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!}, \quad \sec x = \sum_{j=0}^{\infty} (-1)^j E_{2j} \frac{x^{2j}}{(2j)!}, \quad (2)$$

and the Euler numbers  $E_{2j+1} = 0$  for all  $j \geq 0$ .

Our main results are the following theorems.

**Theorem 1.1.** *For  $d \leq n$ ,*

$$T(2n, d) = \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(-1)^j \pi^{2j}}{2^{2d-2} (2j)! d} \binom{2d-2j-2}{d-1} t(2n-2j),$$

where  $t(2j) = 2^{-2j}(2^{2j} - 1)\zeta(2j)$ . Or, equivalently,

$$T(2n, d) = \binom{2d-2}{d-1} \frac{t(2n)}{2^{2d-2}d} - \sum_{j=1}^{\lfloor \frac{d-1}{2} \rfloor} \binom{2d-2j-2}{d-1} \frac{t(2j)t(2n-2j)}{2^{2d-3}(2^{2j}-1)B_{2j}d}.$$

The next three cases after (1) are

$$\begin{aligned} T(2n, 6) &= \frac{21}{512}t(2n) - \frac{7}{192}t(2)t(2n-2) + \frac{1}{256}t(4)t(2n-4), \\ T(2n, 7) &= \frac{33}{1024}\zeta(2n) - \frac{15}{512}t(2)t(2n-2) + \frac{1}{256}t(4)t(2n-4) - \frac{1}{21504}t(6)t(2n-6), \\ T(2n, 8) &= \frac{429}{16384}t(2n) - \frac{99}{4096}t(2)t(2n-2) + \frac{15}{4096}t(4)t(2n-4) - \frac{1}{12288}t(6)t(2n-6). \end{aligned}$$

As we mentioned in the above the proof of Theorem 1.1 utilizes the generating function of  $T(2n, d)$  defined by

$$\Phi(u, v) = 1 + \sum_{n \geq d \geq 1} T(2n, d) u^n v^d$$

for which we have the following result.

**Theorem 1.2.** *We have*

$$\Phi(u, v) = \cos(\pi \sqrt{(1-v)u}/2) \sec(\pi \sqrt{u}/2).$$

The next theorem involves Euler numbers and is more useful computationally when the difference between  $n$  and  $d$  is small.

**Theorem 1.3.** *For  $d \leq n$  we have*

$$T(2n, d) = \frac{(-1)^{n-d} \pi^{2n}}{4^n (2n)!} \sum_{\ell=0}^{n-d} \binom{n-\ell}{d} \binom{2n}{2\ell} E_{2\ell}. \quad (3)$$

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## 2 Proof of Theorem 1.2 and Theorem 1.3

We first recall some results on symmetric functions contained in [1, 2] with some slight modification. Let  $\text{Sym}$  be the subring of  $\mathbb{Q}[[x_1, x_2, \dots]]$  consisting of the formal power series of bounded degree that are invariant under permutations of the  $x_j$ . Define elements  $e_j$ ,  $h_j$ , and  $p_j$  in  $\text{Sym}$  by the generating functions

$$\begin{aligned} E(u) &= \sum_{j=0}^{\infty} e_j u^j = \prod_{j=1}^{\infty} (1 + u x_j), \\ H(u) &= \sum_{j=0}^{\infty} h_j u^j = \prod_{j=1}^{\infty} \frac{1}{1 - u x_j} = E(-u)^{-1}, \\ P(u) &= \sum_{j=1}^{\infty} p_j u^{j-1} = \sum_{j=1}^{\infty} \frac{x_j}{1 - u x_j} = \frac{H'(u)}{H(u)}. \end{aligned}$$

Define a homomorphism  $\mathfrak{T} : \text{Sym} \rightarrow \mathbb{R}$  such that  $\mathfrak{T}(x_j) = 1/(2j-1)^2$  for all  $j \geq 1$ . Hence for all  $n \geq 1$

$$\mathfrak{T}(p_n) = t(2n) = \sum_{j \geq 1} \frac{1}{(2j-1)^{2n}}.$$

First we need a simple lemma.

**Lemma 2.1.** *For any positive integer  $n$  let  $\{2\}^n$  be the string  $(2, \dots, 2)$  with 2 repeated  $n$  times. Then we have*

$$t(\{2\}^n) = \frac{\pi^{2n}}{4^n (2n)!}. \quad (4)$$

*Proof.* It is easy to see that

$$\begin{aligned}
1 + \sum_{n=1}^{\infty} t(\{2\}^n) x^n &= \prod_{j=1}^{\infty} \left( 1 + \frac{x}{(2j-1)^2} \right) \\
&= \prod_{j=1}^{\infty} \left( 1 + \frac{x}{j^2} \right) / \prod_{j=1}^{\infty} \left( 1 + \frac{x}{(2j)^2} \right) \\
&= \frac{\sinh(\pi\sqrt{x})}{\pi\sqrt{x}} \cdot \frac{\pi\sqrt{x}/2}{\sinh(\pi\sqrt{x}/2)} \\
&= \cosh(\pi\sqrt{x}/2) \\
&= \sum_{n=1}^{\infty} \frac{\pi^{2n} x^n}{4^n (2n)!}.
\end{aligned}$$

This finishes the proof of the lemma.  $\square$

Now let  $N_{n,d}$  be the sum of all the monomial symmetric functions corresponding to partitions of  $n$  having length  $d$ . Then clearly

$$\mathfrak{T}(N_{n,d}) = T(2n, d).$$

As in [1] we may define

$$\mathcal{F}(u, v) = 1 + \sum_{n \geq d \geq 1} N_{n,d} u^n v^d,$$

then  $\mathfrak{T}$  sends  $\mathcal{F}(u, v)$  to the generating function

$$\Phi(u, v) = 1 + \sum_{n \geq d \geq 1} T(2n, d) u^n v^d.$$

By Lemma 2.1 we have

$$\mathfrak{T}(e_n) = t(\{2\}^n) = \frac{\pi^{2n}}{4^n (2n)!}. \quad (5)$$

Hence

$$\mathfrak{T}(E(u)) = \cosh(\pi\sqrt{u}/2),$$

and

$$\mathfrak{T}(H(u)) = \mathfrak{T}(E(-u)^{-1}) = 1/\cosh(\pi\sqrt{-u}/2) = \sec(\pi\sqrt{u}/2).$$

Thus by [1, Lemma 1]  $\mathcal{F}(u, v) = E((v-1)u)H(u)$  and we get

$$\begin{aligned}
\Phi(u, v) &= \mathfrak{T}(E((v-1)u)H(u)) = \cosh(\pi\sqrt{(v-1)u}/2) \sec(\pi\sqrt{u}/2) \\
&= \cos(\pi\sqrt{(1-v)u}/2) \sec(\pi\sqrt{u}/2).
\end{aligned}$$

This proves Theorem 1.2.

Setting  $v = 1$  in Theorem 1.2 we obtain

$$\Phi(u, 1) = \sec(\pi\sqrt{u}/2).$$

This yields immediately the following identity by (2)

$$\mathfrak{T}(h_n) = \sum_{d=1}^n T(2n, d) = \frac{(-1)^n E_{2n} \pi^{2n}}{4^n (2n)!}. \quad (6)$$

Now by [1, Lemma 2] we have

$$N_{n,d} = \sum_{\ell=0}^{n-d} \binom{n-\ell}{d} (-1)^{n-d-\ell} h_\ell e_{n-\ell}.$$

Applying the homomorphism  $\mathfrak{T}$  and using equation (4) and (6) we get Theorem 1.3 immediately.

### 3 Proof of Theorems 1.1 and a combinatorial identity

We now rewrite the generating function  $\Phi(4u, v)$  as follows using Theorem 1.2:

$$\Phi(4u, v) = \sum_{d \geq 0} v^d \tilde{G}_d(u) = \sec(\pi\sqrt{u}) \cos(\pi\sqrt{(1-v)u}) = \sec(\pi\sqrt{u}) \sum_{j=0}^{\infty} \frac{\pi^{2j}}{(2j)!} (v-1)^j u^j.$$

Let  $D$  be the differential operator with respect to  $u$ . Then

$$\begin{aligned} \tilde{G}_d(u) &= (-1)^d \sec(\pi\sqrt{u}) \sum_{j \geq d} \frac{(-1)^j \pi^{2j} u^j}{(2j)!} \binom{j}{d} \\ &= \sec(\pi\sqrt{u}) \cdot \frac{(-u)^d}{d!} \cdot D^d \sum_{j \geq d} \frac{(-1)^j \pi^{2j} u^j}{(2j)!} \\ &= \sec(\pi\sqrt{u}) \cdot \frac{(-u)^d}{d!} \cdot D^d \cos(\pi\sqrt{u}) \\ &= -\frac{\pi^2}{2} \sec(\pi\sqrt{u}) \cdot \frac{(-u)^d}{d!} \cdot D^{d-1} \frac{\sin(\pi\sqrt{u})}{\pi\sqrt{u}} \\ &= \frac{\pi^2 u \tan(\pi\sqrt{u})}{2d} \frac{1}{\pi\sqrt{u}} G_{d-1}(u) \end{aligned}$$

by [1, (12)] (the definition of  $G_k$  is defined on page 9). By [1, Lemma 3] we have

$$\tilde{G}_d(u) = -\frac{\pi^2 u}{2d} \sum_{j=0}^{\lfloor \frac{d-2}{2} \rfloor} \frac{(-4\pi^2 u)^j}{2^{2d-3}(2j+1)!} \binom{2d-2j-3}{d-1} \quad (7)$$

$$\begin{aligned} & + \frac{\pi\sqrt{u}}{2d} \tan(\pi\sqrt{u}) \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(-4\pi^2 u)^j}{2^{2d-2}(2j)!} \binom{2d-2j-2}{d-1} \\ & = \frac{\pi\sqrt{u}}{2d} \tan(\pi\sqrt{u}) \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(-4\pi^2 u)^j}{2^{2d-2}(2j)!} \binom{2d-2j-2}{d-1} + \text{terms of degree } < d. \end{aligned} \quad (8)$$

It is well-known that

$$\tan x = \sum_{m=1}^{\infty} \frac{(-1)^{m-1} 2^{2m} (2^{2m} - 1) B_{2m} x^{2m-1}}{(2m)!}.$$

Hence

$$\frac{\pi\sqrt{u}}{2} \tan(\pi\sqrt{u}) = \sum_{m=1}^{\infty} 4^m t(2m) u^m.$$

Therefore  $T(2n, d)$  is the coefficient of  $u^n$  in

$$\tilde{G}_d(u/4) = \frac{1}{d} \sum_{m=2}^{\infty} t(2m) u^m \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(-\pi^2 u)^j}{2^{2d-2}(2j)!} \binom{2d-2j-2}{d-1}.$$

This implies Theorem 1.1 immediately. Notice that by comparing Theorem 1.1 and Theorem 1.3 we get the following identity of between Bernoulli numbers and Euler numbers.

**Theorem 3.1.** *For all  $d \leq n$*

$$\sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(2^{2n-2j} - 1) B_{2n-2j}}{2^{2d-1} d} \binom{2d-2j-2}{d-1} \binom{2n}{2j} = \frac{(-1)^{n-d} \pi^{2n}}{4^n (2n)!} \sum_{\ell=0}^{n-d} \binom{n-\ell}{d} \binom{2n}{2\ell} E_{2\ell}.$$

Further we have

$$\begin{aligned} & \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(2^{2n-2j} - 1) B_{2n-2j}}{2^{2d-1} d} \binom{2d-2j-2}{d-1} \binom{2n}{2j} \\ & = \begin{cases} 0, & \text{if } n < d < 2n; \\ \frac{n}{2^{2d-1} d} \binom{2d-2n-1}{d-1}, & \text{if } d \geq 2n. \end{cases} \end{aligned}$$

*Proof.* We only need to show the second identity. Notice that when  $d > n$  the coefficient of  $u^n v^d$  is 0 in  $\Phi(u, v)$ . Thus the coefficient of  $u^n$  in  $\tilde{G}_d(u/4)$  is zero. By (7) and (8) we have

$$\begin{aligned}
& \sum_{j=0}^{\lfloor \frac{d-1}{2} \rfloor} \frac{(2^{2n-2j} - 1)B_{2n-2j}}{2^{2d-1}d} \binom{2d-2j-2}{d-1} \binom{2n}{2j} \\
&= \frac{(-1)^n (2n)!}{(2\pi)^{2n}} \times \text{Coeff. of } u^n \text{ of (7) (i.e. } j = n-1) \\
&= \begin{cases} 0, & n < d < 2n; \\ \frac{n}{2^{2d-1}d} \binom{2d-2n-1}{d-1}, & d \geq 2n, \end{cases}
\end{aligned}$$

as desired. □

## References

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